S.-T. Yau College Student Mathematics Contests 2022

Analysis and differential equations individual

Problem 1. Assume U is a bounded smooth open set, $q \in [1,\infty)$ and $f_k \rightharpoonup f$ weakly in $L^q(U)$ and $f_k \to f$ a.e in U, then $\lim_{k\to\infty}\left(\|f_k\|_{L^q(U)} - \|f_k - f\|_{L^q(U)}\right) = \|f\|_{L^q(U)}$.

Problem 2. Let Ω be a proper (nonempty and $\neq \mathbb{C}$) region of $\mathbb{C}$ which is simply connected. Let $\mathbb{D}$ be the unit disc and $z_0 \in \Omega$, and

$$\mathcal{F} = \{f | f : \Omega \to \mathbb{D} \text{ holomorphic, injective and } f(z_0) = 0\}$$

One strategy of proving the (existence part of the) Riemann mapping theorem is that the desired map f satisfies

$$f'(z_0) = \sup_{g\in\mathcal{F}} |g'(z_0)|.$$

Try to explain the proof following this strategy as detailed as possible.

Problem 3. Assume that u solves the nonlinear heat equation

$u_t = \frac{u_{xx}}{u_x^2}$ in $\mathbf{R} \times (0,\infty)$

with $u_x > 0$. Find a transformation which changes the above equation into a linear PDE.

Problem 4. Let $\phi \in C^\infty([0,T],\mathbb{R}^n)$ be a solution of linear wave equation

$$\sum_{\alpha,\beta=0}^{n} g^{\alpha\beta}\partial_\alpha\partial_\beta\phi = F,$$

where $\partial_\alpha := \frac{\partial}{\partial x_\alpha}, \alpha = 0,1,2,\cdots,n,\ t = x_0,\ g^{\alpha\beta}, F$ are smooth functions in $[0,T]\times\mathbb{R}^n$, $g^{\alpha\beta}$ is symmetric and

$$\sup_{[0,T]\times\mathbb{R}^n} \sum_{\alpha,\beta} |g^{\alpha\beta} - \eta^{\alpha\beta}| < \frac{1}{2022},$$

where $\eta_{00} = -1, \eta_{ii} = 1, i = 1,2,\cdots,n$ and $\eta_{\alpha i} = 0$ otherwise. Show that there is a constant C depending only on n such that for any $t \in [0,T]$,

$$\|\partial\phi(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le C\left(\|\partial\phi(0,\cdot)\|_{L^2(\mathbb{R}^n)} + \int_0^t \|F(s,\cdot)\|_{L^2(\mathbb{R}^n)}\mathrm{d}s\right)$$
$$\times \exp\left(\int_0^t \sum_{\alpha,\beta} \|\partial g^{\alpha\beta}(s,\cdot)\|_{L^2(\mathbb{R}^n)}\mathrm{d}s\right),$$

here $|\partial f|^2 := \sum_{\alpha=0}^n |\partial_\alpha f|^2$.